Bases para un espacio de funciones

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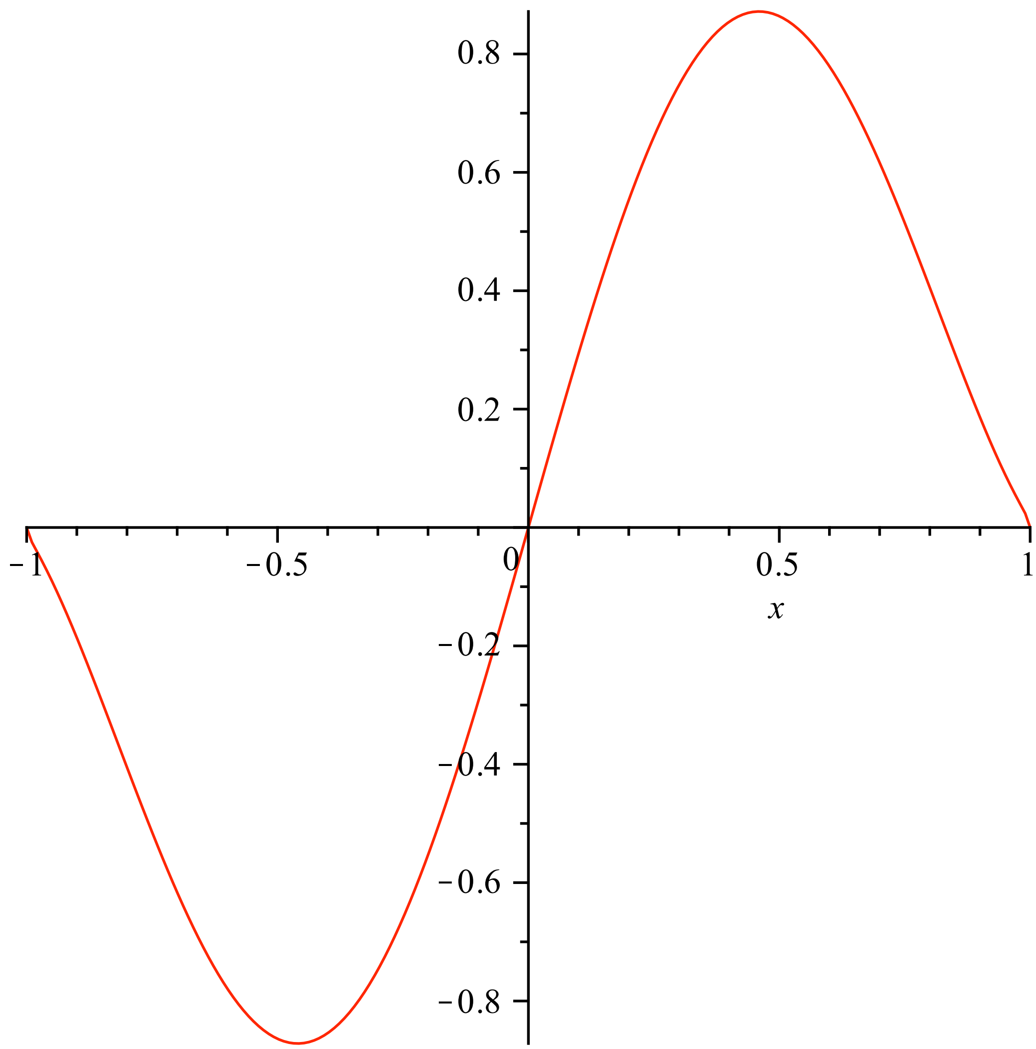
```
> restart: with(plots):
```

Consideremos una función sencilla

```
> f := x -> sqrt(1-x^2)*sin(3*x);
```

$$f := x \rightarrow \sqrt{1 - x^2}\,\sin(3\,x) \quad (1)$$

```
> plot(f(x),x=-1..1);
```



y recordemos la aproximación por serie de Taylor

```
> n:=8;
```

$$n := 8 \quad (2)$$

```
> Taylor[n]:= taylor(f(x),x=0,n);
```

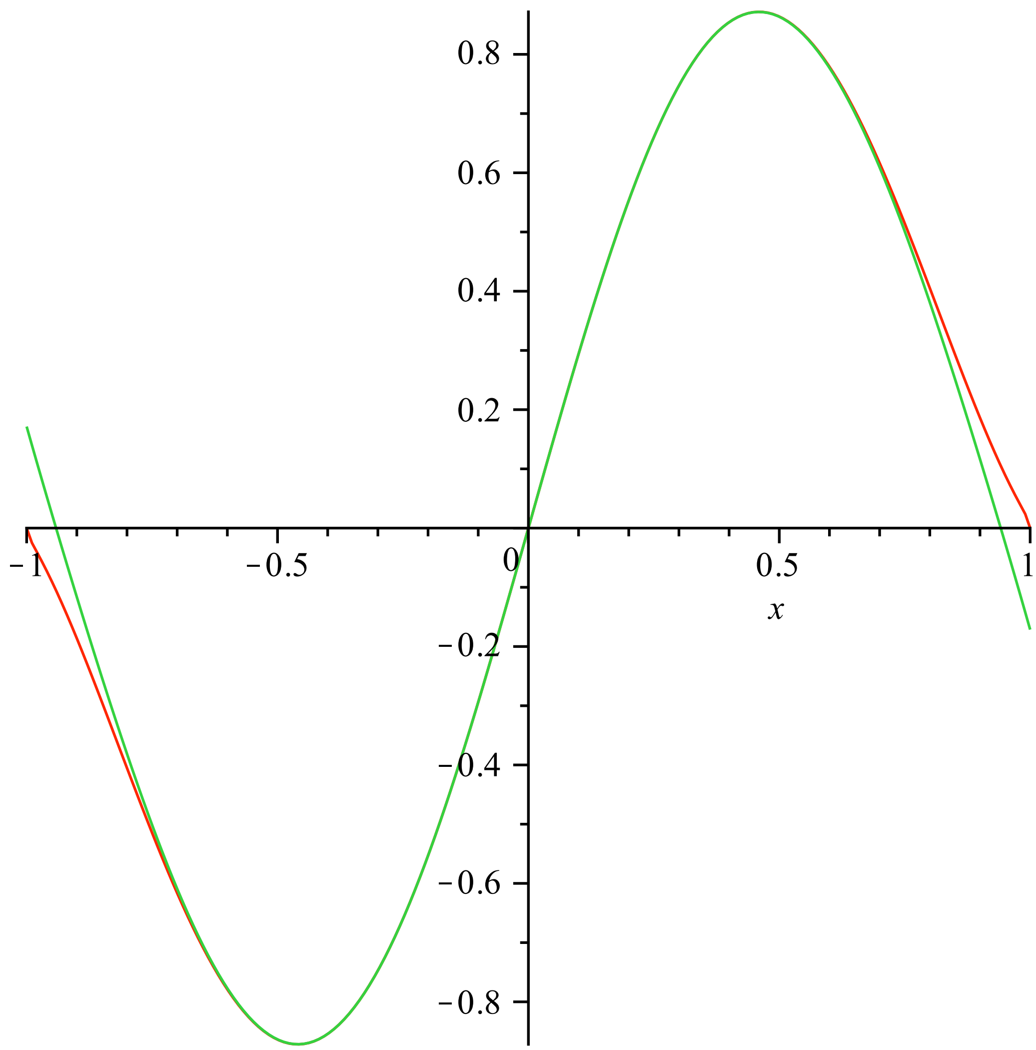
$$Taylor_8 := 3\,x - 6\,x^3 + \frac{39}{10}\,x^5 - \frac{15}{14}\,x^7 + \mathrm{O}\left(x^8\right) \qquad (3)$$

```
> AproxTaylor[n]:=convert(Taylor[n],polynom);
```

$$AproxTaylor_8 := 3\,x - 6\,x^3 + \frac{39}{10}\,x^5 - \frac{15}{14}\,x^7 \qquad (4)$$

La serie de Taylor no es más que una expansión en una base de monomios en la cual conocemos los coeficientes y sabemos como generarlos. Si graficamos la función y su aproximación vemos

```
> plot([f(x),AproxTaylor[n]],x=-1..1);
```



El producto interno en esta base de monomios lo podemos definir como

```
> Int(x^l*x^m,x=-1..1)=int(x^l*x^m,x=-1..1);
```

$$\int_{-1}^{1} x^l\,x^m\,\mathrm{d}x = \frac{1 + (-1)^{l+m}}{l+m+1} \qquad (5)$$

Claramente no es ortogonal

```
> Prodintern:=int(x^l*x^m,x=-1..1);subs({l=3,m=2},Prodintern);
```

$$Prodintern := \frac{1 + (-1)^{l+m}}{l+m+1}$$

(6)

$$0 \quad (6)$$

El error de la aproximación será la norma de la resta de los dos vectores. Esto es

```
> ErrorTaylor := sqrt(int((f(x)-AproxTaylor[n])^2,x=-1..1));
```

ErrorTaylor :=

$$\frac{1}{20790}\left(308314512+72037350\sqrt{\pi}\left(\frac{4}{\sqrt{\pi}}+\frac{1}{3}\frac{\cos(6)}{\sqrt{\pi}}-\frac{1}{18}\frac{\sin(6)}{\sqrt{\pi}}\right)\right.$$
$$-864448200\,\pi\left(-\mathrm{BesselJ}(0,3)+\frac{2}{3}\mathrm{BesselJ}(1,3)\right)+768398400\,\pi\left(\frac{3}{4}\mathrm{BesselJ}(0,3)+\frac{7}{4}\mathrm{BesselJ}(1,3)\right)-221981760\,\pi\left(-\frac{45}{8}\mathrm{BesselJ}(0,3)-3\,\mathrm{BesselJ}(1,3)\right)$$
$$\left.+27104000\,\pi\left(\frac{1233}{16}\mathrm{BesselJ}(0,3)+\frac{1947}{32}\mathrm{BesselJ}(1,3)\right)\right)^{1/2} \quad (7)$$

```
> evalf(%);
```

$$0.05870966633 \quad (8)$$

```
> restart: with(plots): with(orthopoly);
```

$$[G, H, L, P, T, U] \quad (9)$$

Los polinomios de Legendre son polinomios definidos en el intervalo -1..1
En MAPLE se denotan como P(n,x) donde *n* es el orden del polinomo y *x* la variable. Para detalles pueden consultar la hoja de ayuda

```
> ?P
> P(0,x);P(1,x);P(2,x);P(3,x);P(4,x);P(5,x);
```

$$1$$
$$x$$
$$-\frac{1}{2}+\frac{3}{2}x^2$$
$$\frac{5}{2}x^3-\frac{3}{2}x$$
$$\frac{3}{8}+\frac{35}{8}x^4-\frac{15}{4}x^2$$
$$\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x \quad (10)$$

Los Polinomios de Legendre son ortogonales bajo un producto interno definido por $< P_n \mid P_m > = \int_{-1}^{1} P_n P_m \, dx$

```
> int(P(n,x)*P(m,x),x=-1..1);
```

$$\int_{-1}^{1} P(n,x)\,P(m,x)\,\mathrm{d}x \quad (11)$$

```
> int(P(13,x)*P(28,x),x=-1..1);
```

$$0 \quad (12)$$

```
> int(P(77,x)*P(77,x),x=-1..1);
```

(13)

$$\frac{2}{155} \quad (13)$$

```
> f := x -> sqrt(1-x^2)*sin(3*x);
```

$$f := x \rightarrow \sqrt{1-x^2}\,\sin(3\,x) \quad (14)$$

Por lo tanto los coeficientes de la expansión serán $< f \mid P_m > = \int_{-1}^{1} f(x)\, P_m(x)\, dx$

m=0

```
> Int(f(x)*P(0,x),x=-1..1)/Int(P(0,x)*P(0,x),x=-1..1)=int(f(x)*P(0,
x),x=-1..1)/int(P(0,x)*P(0,x),x=-1..1);
```

$$\frac{\int_{-1}^{1} \sqrt{1-x^2}\,\sin(3\,x)\,dx}{\int_{-1}^{1} 1\,dx} = 0 \quad (15)$$

m=1

```
> Int(f(x)*P(1,x),x=-1..1)/Int(P(1,x)*P(1,x),x=-1..1)=int(f(x)*P(1,
x),x=-1..1)/int(P(1,x)*P(1,x),x=-1..1);
```

$$\frac{\int_{-1}^{1} \sqrt{1-x^2}\,\sin(3\,x)\,x\,dx}{\int_{-1}^{1} x^2\,dx} = \frac{1}{2}\,\pi\left(-\mathrm{BesselJ}(0,3) + \frac{2}{3}\,\mathrm{BesselJ}(1,3)\right) \quad (16)$$

```
> evalf(%);
```

$$0.7635503665 = 0.7635503665 \quad (17)$$

m=2

```
> Int(f(x)*P(2,x),x=-1..1)/Int(P(2,x)*P(2,x),x=-1..1)=int(f(x)*P(2,
x),x=-1..1)/int(P(2,x)*P(2,x),x=-1..1);
```

$$\frac{\int_{-1}^{1} \sqrt{1-x^2}\,\sin(3\,x)\left(-\frac{1}{2}+\frac{3}{2}\,x^2\right)dx}{\int_{-1}^{1}\left(-\frac{1}{2}+\frac{3}{2}\,x^2\right)^2 dx} = 0 \quad (18)$$

m=3

```
> Int(f(x)*P(3,x),x=-1..1)/Int(P(3,x)*P(3,x),x=-1..1)=int(f(x)*P(3,
x),x=-1..1)/int(P(3,x)*P(3,x),x=-1..1);
```

$$\frac{\int_{-1}^{1} \sqrt{1-x^2}\,\sin(3\,x)\left(\frac{5}{2}\,x^3-\frac{3}{2}\,x\right)dx}{\int_{-1}^{1}\left(\frac{5}{2}\,x^3-\frac{3}{2}\,x\right)^2 dx} = \frac{35}{27}\,\pi\left(\frac{3}{4}\,\mathrm{BesselJ}(0,3) + \frac{7}{4}\,\mathrm{BesselJ}(1,3)\right) \quad (19)$$

$$-\frac{7}{4}\pi\left(-\text{BesselJ}(0,3)+\frac{2}{3}\text{BesselJ}(1,3)\right)$$

```
> evalf(%);
```

$$-1.050317580=-1.050317580 \quad (20)$$

m=4

```
> Int(f(x)*P(4,x),x=-1..1)/Int(P(4,x)*P(4,x),x=-1..1)=int(f(x)*P(4,x),x=-1..1)/int(P(4,x)*P(4,x),x=-1..1);
```

$$\frac{\int_{-1}^{1}\sqrt{1-x^2}\,\sin(3x)\left(\frac{3}{8}+\frac{35}{8}x^4-\frac{15}{4}x^2\right)dx}{\int_{-1}^{1}\left(\frac{3}{8}+\frac{35}{8}x^4-\frac{15}{4}x^2\right)^2dx}=0 \quad (21)$$

en general los coeficientes de la expansión se expresan

```
> CoefLegendre[k]:= int(f(x)*P(k,x),x=-1..1)/int(P(k,x)*P(k,x),x=-1..1);
```

$$CoefLegendre_k:=\frac{\int_{-1}^{1}\sqrt{1-x^2}\,\sin(3x)\,P(k,x)\,dx}{\int_{-1}^{1}P(k,x)^2\,dx} \quad (22)$$

y la aproximación de la función

```
> f(x)=Sum(CoefLegendre[k]*P(k,x),k=0..n);
```

$$\sqrt{1-x^2}\,\sin(3x)=\sum_{k=0}^{n}\frac{\left(\int_{-1}^{1}\sqrt{1-x^2}\,\sin(3x)\,P(k,x)\,dx\right)P(k,x)}{\int_{-1}^{1}P(k,x)^2\,dx} \quad (23)$$

```
> n:=7;
```

$$n:=7 \quad (24)$$

```
> f(x)=Sum(CoefLegendre[k]*P(k,x),k=0..n);
```

$$\sqrt{1-x^2}\,\sin(3x)=\sum_{k=0}^{7}\frac{\left(\int_{-1}^{1}\sqrt{1-x^2}\,\sin(3x)\,P(k,x)\,dx\right)P(k,x)}{\int_{-1}^{1}P(k,x)^2\,dx} \quad (25)$$

```
> AproxLegendre[n]:= sum(CoefLegendre[k]*P(k,x),k=0..n);
```

$$AproxLegendre_7:=\frac{1}{2}\pi\left(-\text{BesselJ}(0,3)+\frac{2}{3}\text{BesselJ}(1,3)\right)x \quad (26)$$

$$+\frac{7}{2}\left(\frac{10}{27}\pi\left(\frac{3}{4}\text{BesselJ}(0,3)+\frac{7}{4}\text{BesselJ}(1,3)\right)-\frac{1}{2}\pi\left(-\text{BesselJ}(0,3)\right.\right.$$

$$\left.\left.+\frac{2}{3}\text{BesselJ}(1,3)\right)\right)\left(\frac{5}{2}x^3-\frac{3}{2}x\right)+\frac{11}{2}\left(\frac{14}{27}\pi\left(-\frac{45}{8}\text{BesselJ}(0,3)\right.\right.$$

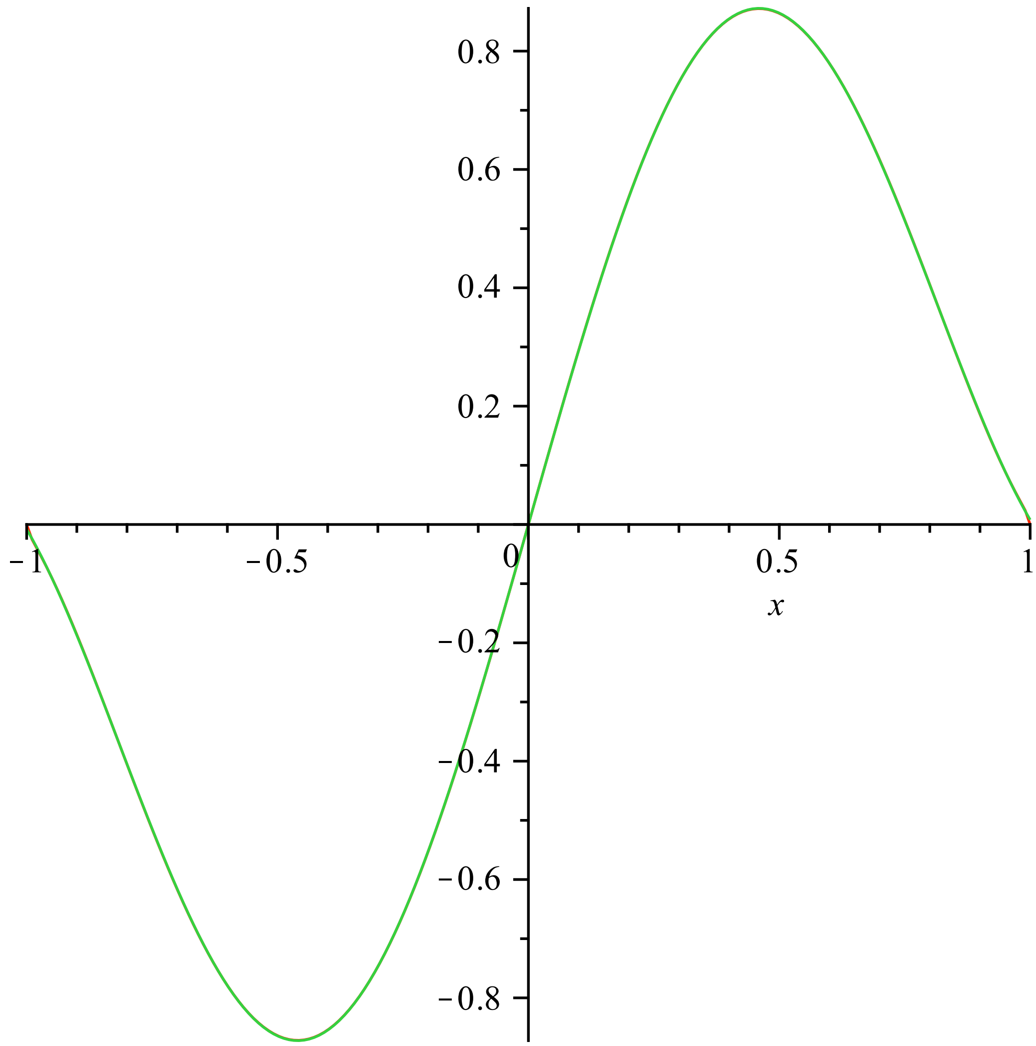
$$- 3\,\mathrm{BesselJ}(1,3)\Big) - \frac{35}{27}\pi\left(\frac{3}{4}\mathrm{BesselJ}(0,3) + \frac{7}{4}\mathrm{BesselJ}(1,3)\right) + \frac{5}{8}\pi\Big(-\mathrm{BesselJ}(0,3) + \frac{2}{3}\mathrm{BesselJ}(1,3)\Big)\Big)\left(\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x\right)$$

$$+ \frac{15}{2}\left(\frac{572}{729}\pi\left(\frac{1233}{16}\mathrm{BesselJ}(0,3) + \frac{1947}{32}\mathrm{BesselJ}(1,3)\right) - \frac{77}{27}\pi\left(-\frac{45}{8}\mathrm{BesselJ}(0,3) - 3\,\mathrm{BesselJ}(1,3)\right) + \frac{35}{12}\pi\left(\frac{3}{4}\mathrm{BesselJ}(0,3) + \frac{7}{4}\mathrm{BesselJ}(1,3)\right) - \frac{35}{48}\pi\left(-\mathrm{BesselJ}(0,3) + \frac{2}{3}\mathrm{BesselJ}(1,3)\right)\right)\left(\frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x\right)$$

```
> simplify(AproxLegendre[n]);
```

$$\frac{1}{41472}\pi x\,(-56160930\,\mathrm{BesselJ}(0,3) - 42957775\,\mathrm{BesselJ}(1,3) + 495654390\,\mathrm{BesselJ}(0,3)\,x^2 + 379926855\,\mathrm{BesselJ}(1,3)\,x^2 - 1078107030\,\mathrm{BesselJ}(0,3)\,x^4 - 826746921\,\mathrm{BesselJ}(1,3)\,x^4 + 662380290\,\mathrm{BesselJ}(0,3)\,x^6 + 508006785\,\mathrm{BesselJ}(1,3)\,x^6) \qquad \textbf{(27)}$$

```
> plot([f(x),AproxLegendre[n]],x=-1..1);
```



Otra vez, el error de la aproximación será la norma de la distancia entre los dos vectores. Esto es

```
> ErrorLegendre := sqrt(int((f(x)-AproxLegendre[n])^2,x=-1..1));
```

$$ErrorLegendre := \frac{1}{7776}\left(-2867674412880\,\pi^2\,\mathrm{BesselJ}(0,3)^2 - 4398390551160\,\pi^2\,\mathrm{BesselJ}(0,3)\,\mathrm{BesselJ}(1,3) - 1686582759198\,\pi^2\,\mathrm{BesselJ}(1,3)^2 + 40310784 + 3359232\cos(6) - 559872\sin(6)\right)^{1/2} \quad \textbf{(28)}$$

es decir

```
> evalf(%);
```

$$0.002386344815\,I \quad \textbf{(29)}$$

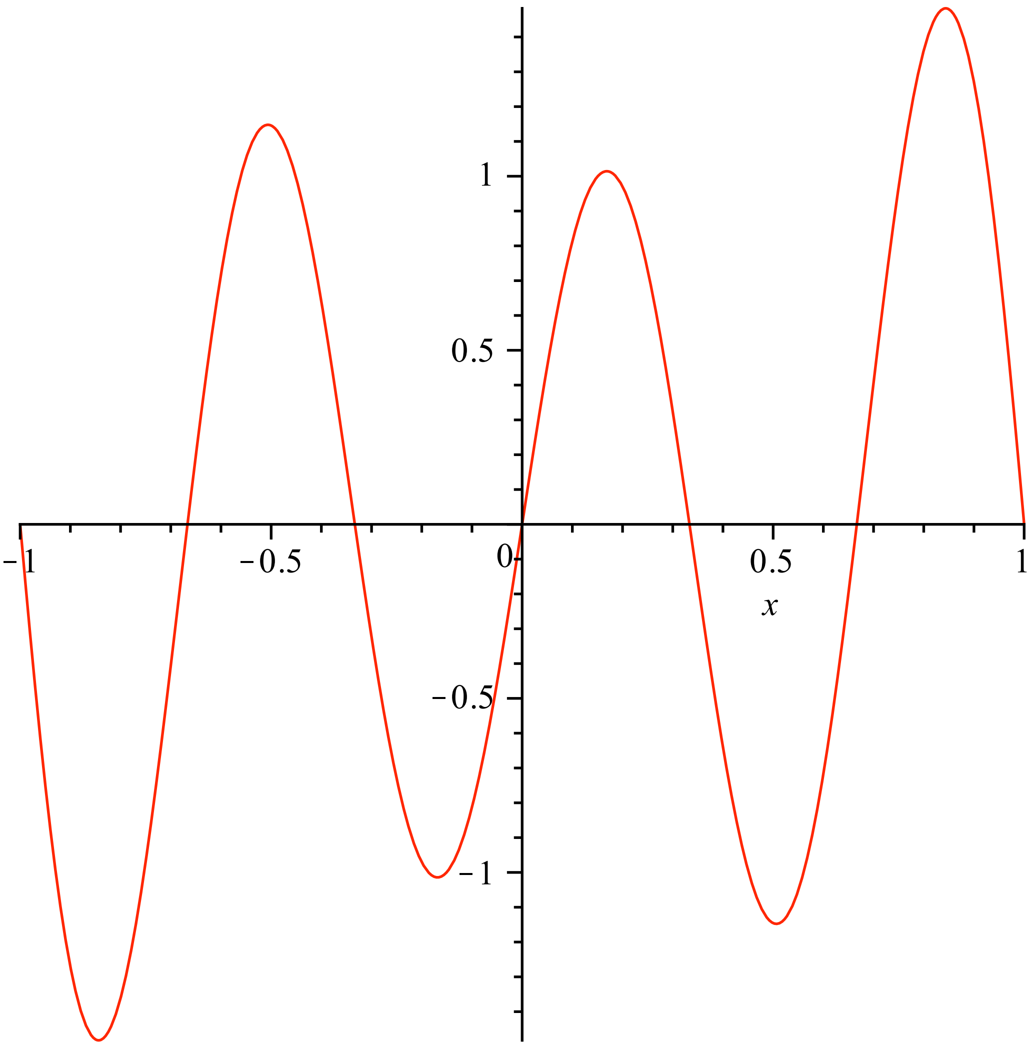
Si seleccionamos una función un poco mas compleja

```
> restart: with(plots): with(orthopoly);
```

$$[G, H, L, P, T, U] \quad \textbf{(30)}$$

```
> f := x -> sqrt(x^4+x^2+1)*sin(3*Pi*x);plot(f(x), x=-1..1);
```

$$f := x \to \sqrt{x^4 + x^2 + 1}\,\sin(3\,\pi\,x)$$



Probamos con Taylor

```
> n:=10;
```

$$n := 10 \tag{31}$$

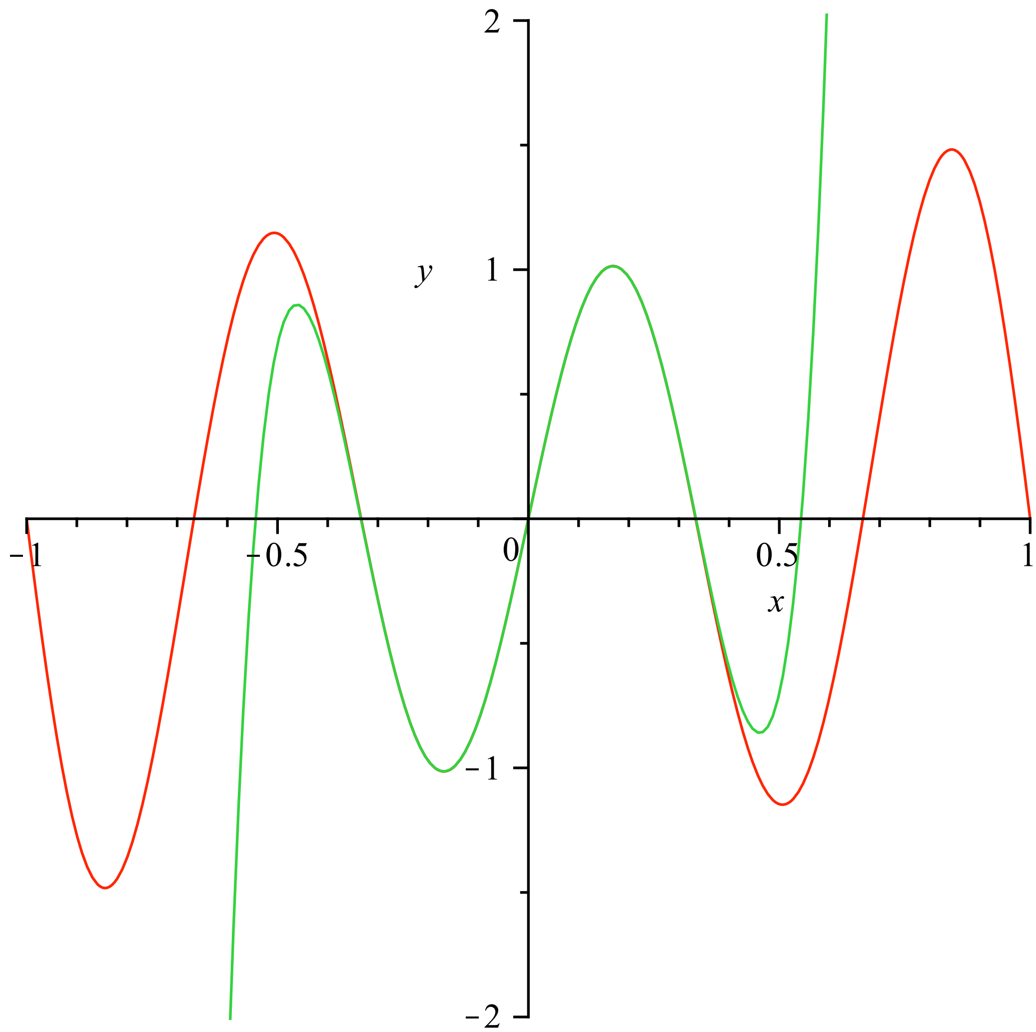
```
> Taylor[n]:= taylor(f(x),x=0,n);
```

$$Taylor_{10} := 3\pi x + \left(-\frac{9}{2}\pi^3 + \frac{3}{2}\pi\right)x^3 + \left(\frac{81}{40}\pi^5 + \frac{9}{8}\pi - \frac{9}{4}\pi^3\right)x^5 + \left(-\frac{27}{16}\pi^3 - \frac{243}{560}\pi^7 - \frac{9}{16}\pi + \frac{81}{80}\pi^5\right)x^7 + \left(\frac{243}{4480}\pi^9 + \frac{27}{32}\pi^3 + \frac{243}{320}\pi^5 - \frac{243}{1120}\pi^7 + \frac{9}{128}\pi\right)x^9 + O\left(x^{10}\right) \tag{32}$$

```
> AproxTaylor[n]:=convert(Taylor[n],polynom);
```

$$AproxTaylor_{10} := 3\pi x + \left(-\frac{9}{2}\pi^3 + \frac{3}{2}\pi\right)x^3 + \left(\frac{81}{40}\pi^5 + \frac{9}{8}\pi - \frac{9}{4}\pi^3\right)x^5 + \left(-\frac{27}{16}\pi^3 - \frac{243}{560}\pi^7 - \frac{9}{16}\pi + \frac{81}{80}\pi^5\right)x^7 + \left(\frac{243}{4480}\pi^9 + \frac{27}{32}\pi^3 + \frac{243}{320}\pi^5 - \frac{243}{1120}\pi^7 + \frac{9}{128}\pi\right)x^9 \tag{33}$$

```
> plot([f(x),AproxTaylor[n]],x=-1..1,y=-2..2);
```



y con Legendre

```
> CoefLegendre[k]:= int(f(x)*P(k,x),x=-1..1)/int(P(k,x)*P(k,x),x=
-1..1);
```

$$CoefLegendre_k := \frac{\int_{-1}^{1} \sqrt{x^4 + x^2 + 1}\, \sin(3\,\pi\,x)\, P(k, x)\, dx}{\int_{-1}^{1} P(k, x)^2\, dx} \tag{34}$$

```
> AproxLegendre[n]:= sum(CoefLegendre[k]*P(k,x),k=0..n);
```

$$AproxLegendre_{10} := \frac{3}{2}\left(\int_{-1}^{1} \sqrt{x^4 + x^2 + 1}\, \sin(3\,\pi\,x)\, x\, dx\right) x + \frac{7}{2}\left(\int_{-1}^{1} \sqrt{x^4 + x^2 + 1}\, \sin(3\,\pi\,x)\left(\frac{5}{2}\,x^3 - \frac{3}{2}\,x\right) dx\right)\left(\frac{5}{2}\,x^3 - \frac{3}{2}\,x\right) + \frac{11}{2}\left(\right. \tag{35}$$

$$\int_{-1}^{1} \sqrt{x^4+x^2+1}\,\sin(3\pi x)\left(\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x\right)dx\Bigg)\left(\frac{63}{8}x^5-\frac{35}{4}x^3\right.$$

$$\left.+\frac{15}{8}x\right)+\frac{15}{2}\left(\left(\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\left(\frac{429}{16}x^7-\frac{693}{16}x^5+\frac{315}{16}x^3\right.\right.\right.$$

$$\left.\left.\left.-\frac{35}{16}x\right)dx\right)\left(\frac{429}{16}x^7-\frac{693}{16}x^5+\frac{315}{16}x^3-\frac{35}{16}x\right)+\frac{19}{2}\Bigg(\right.$$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\left(\frac{12155}{128}x^9-\frac{6435}{32}x^7+\frac{9009}{64}x^5-\frac{1155}{32}x^3\right.$$

$$\left.\left.+\frac{315}{128}x\right)dx\right)\left(\frac{12155}{128}x^9-\frac{6435}{32}x^7+\frac{9009}{64}x^5-\frac{1155}{32}x^3+\frac{315}{128}x\right)$$

```
> simplify(%);
```

$$\frac{1}{32768}x\Bigg(49152\left(\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,dx\right)+143360\Bigg(\tag{36}$$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(5x^2-3)\,dx\Bigg)x^2-86016\Bigg($$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(5x^2-3)\,dx\Bigg)+177408\Bigg($$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(63x^4-70x^2+15)\,dx\Bigg)x^4-197120\Bigg($$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(63x^4-70x^2+15)\,dx\Bigg)x^2+42240\Bigg($$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(63x^4-70x^2+15)\,dx\Bigg)+411840\Bigg($$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(429x^6-693x^4+315x^2-35)\,dx\Bigg)x^6-665280\Bigg($$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(429x^6-693x^4+315x^2-35)\,dx\Bigg)x^4+302400\Bigg($$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(429x^6-693x^4+315x^2-35)\,dx\Bigg)x^2-33600\Bigg($$

$$\int_{-1}^{1}\sqrt{x^4+x^2+1}\,\sin(3\pi x)\,x\,(429x^6-693x^4+315x^2-35)\,dx\Bigg)+230945\Bigg($$

$$\int_{-1}^{1} \sqrt{x^4+x^2+1}\ \sin(3\pi x)\ x\left(12155x^8-25740x^6+18018x^4-4620x^2+315\right)\mathrm{d}x\Bigg)$$

$$x^8-489060\left(\left(\int_{-1}^{1} \sqrt{x^4+x^2+1}\ \sin(3\pi x)\ x\left(12155x^8-25740x^6+18018x^4\right.\right.\right.$$

$$\left.\left.-4620x^2+315\right)\mathrm{d}x\right)x^6+342342\left(\left(\int_{-1}^{1} \sqrt{x^4+x^2+1}\ \sin(3\pi x)\ x\left(12155x^8\right.\right.\right.$$

$$\left.\left.-25740x^6+18018x^4-4620x^2+315\right)\mathrm{d}x\right)x^4-87780\Bigg($$

$$\int_{-1}^{1} \sqrt{x^4+x^2+1}\ \sin(3\pi x)\ x\left(12155x^8-25740x^6+18018x^4-4620x^2+315\right)\mathrm{d}x\Bigg)$$

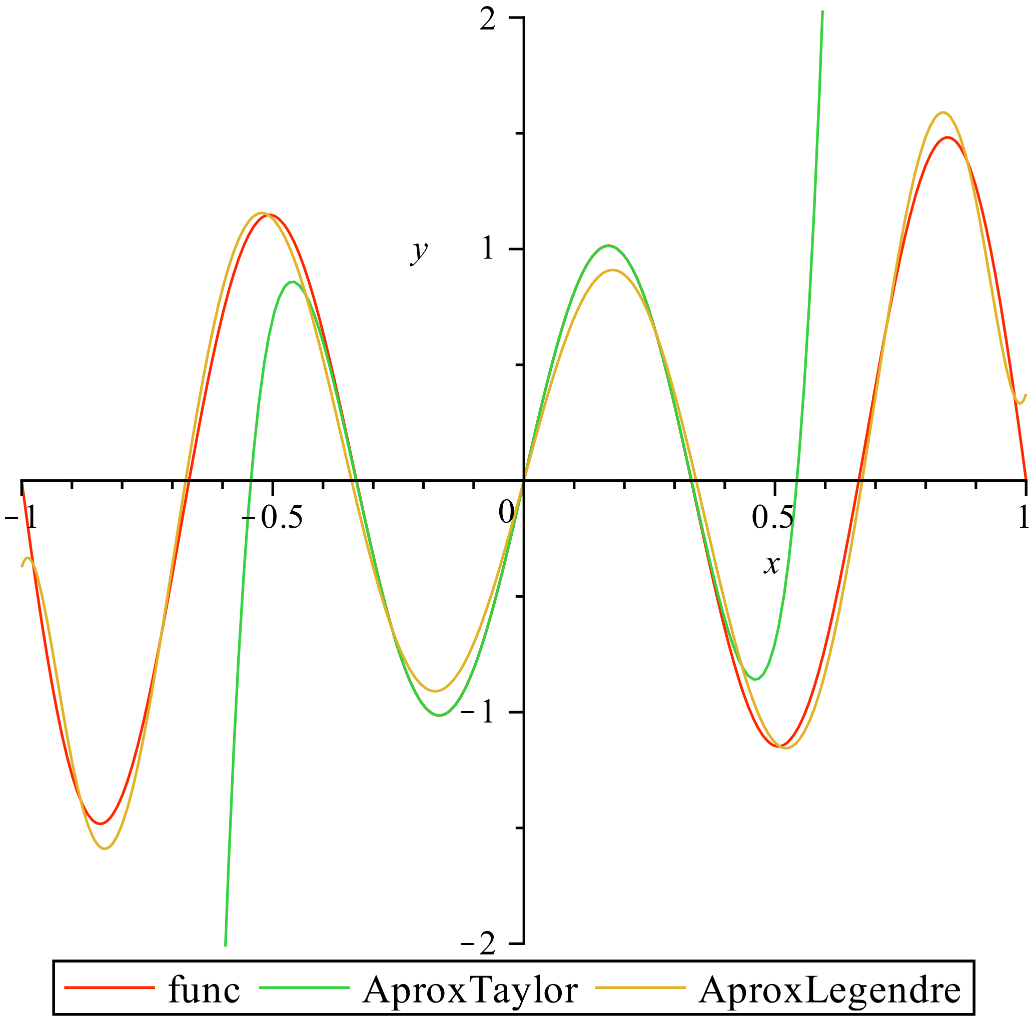
$$x^2+5985\left(\left(\int_{-1}^{1} \sqrt{x^4+x^2+1}\ \sin(3\pi x)\ x\left(12155x^8-25740x^6+18018x^4-4620x^2\right.\right.\right.$$

$$\left.\left.\left.+315\right)\mathrm{d}x\right)\right)$$

```
> AproxLegendreDefitiva:=evalf(%);
```

$$AproxLegendreDefitiva := 0.00003051757812\ x\left(2.628868717\ 10^5-3.343829453\ 10^6\ x^2\right.$$
$$\left.+1.087574358\ 10^7\ x^4-1.283249441\ 10^7\ x^6+5.049952524\ 10^6\ x^8\right) \quad \textbf{(37)}$$

```
> plot([f(x),AproxTaylor[n],AproxLegendreDefitiva],x=-1..1,y=-2..2,
legend=["func","AproxTaylor","AproxLegendre"]);
```



Los errores

```
> ErrorTaylor := sqrt(int((f(x)-AproxTaylor[n])^2,x=-1..1));
```

ErrorTaylor := **(38)**

$$\left(\int_{-1}^{1}\left(\sqrt{x^4+x^2+1}\,\sin(3\,\pi\,x)-3\,\pi\,x-\left(-\frac{9}{2}\,\pi^3+\frac{3}{2}\,\pi\right)x^3-\left(\frac{81}{40}\,\pi^5+\frac{9}{8}\,\pi-\frac{9}{4}\,\pi^3\right)x^5-\left(-\frac{27}{16}\,\pi^3-\frac{243}{560}\,\pi^7-\frac{9}{16}\,\pi+\frac{81}{80}\,\pi^5\right)x^7-\left(\frac{243}{4480}\,\pi^9+\frac{27}{32}\,\pi^3+\frac{243}{320}\,\pi^5-\frac{243}{1120}\,\pi^7+\frac{9}{128}\,\pi\right)x^9\right)^2\mathrm{d}x\right)^{1/2}$$

```
> evalf(%);
```

182.0489197 **(39)**

```
> ErrorLegendre := sqrt(int((f(x)-AproxLegendreDefitiva)^2,x=-1..1)
  );
```

ErrorLegendre := **(40)**

$$\left(\int_{-1}^{1}\left(\sqrt{x^4+x^2+1}\,\sin(3\,\pi\,x)-0.00003051757812\,x\,(2.628868717\,10^5\right.\right.$$

$$-3.343829453\,10^6\,x^2+1.087574358\,10^7\,x^4-1.283249441\,10^7\,x^6$$

$$\left.\left.+5.049952524\,10^6\,x^8)\right)^2\,\mathrm{d}x\right)^{1/2}$$

```
> evalf(%);
```

$$0.1288678040 \tag{41}$$