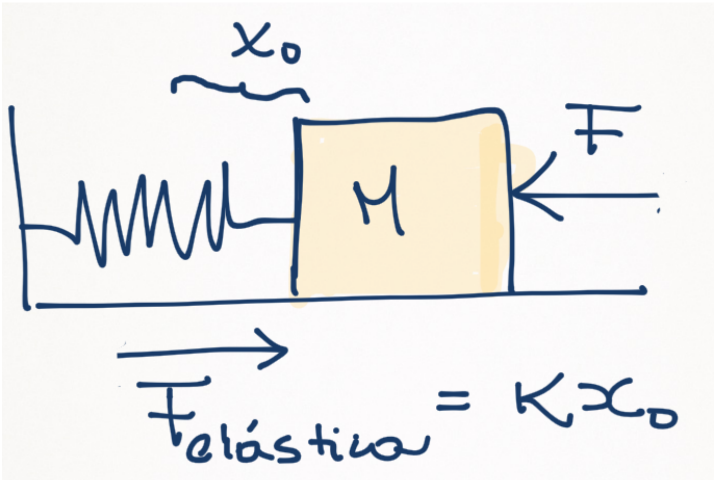
Las fuerzas elásticas
y el
Movimiento oscilatorio

Luis A. Núñez

Esc. Física

Universidad Industrial de Santander

Fuerzas elásticas

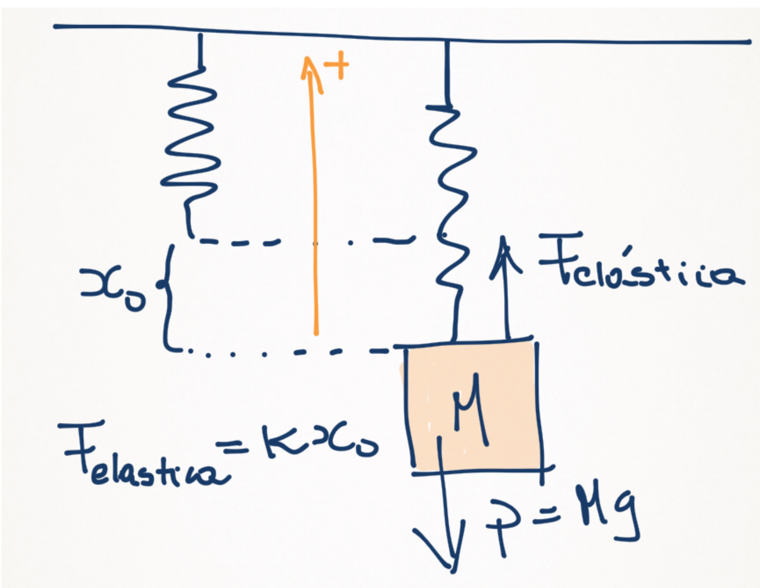


$$-k\vec{\Delta x} = \vec{F}$$

La fuerza del resorte se opone al desplazamiento

$$\sum_{i=1}^{N} \vec{F}_i^{Ext} = 0$$

$$kx_0 - F = 0 \qquad \Rightarrow F = kx_0$$

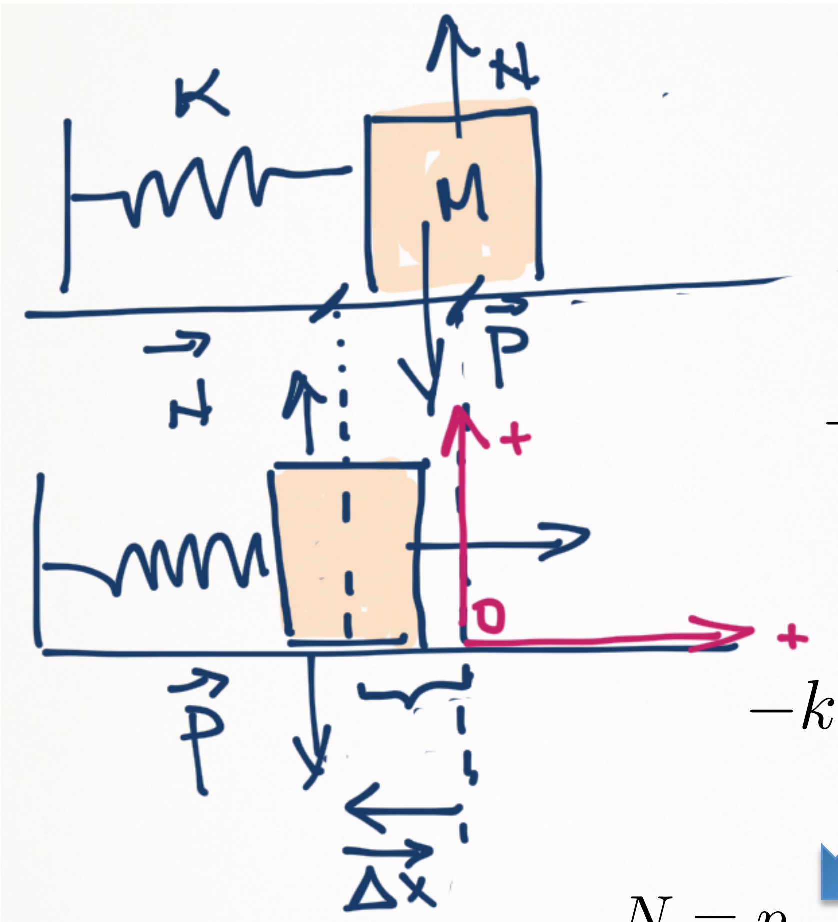


$$\sum_{i=1}^{N} \vec{F}_i^{Ext} = 0$$

$$kx_0 - p = 0 \qquad \Rightarrow p = kx_0$$

$$x_0 = \frac{p}{k}$$

Para un resorte



$$\vec{N} + \vec{p} - k\vec{\Delta x} = m\vec{a}$$

$$N - p = 0$$

$$-k(x - x_0) = m a_x$$

$$N - p = 0$$

$$-kx = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$N = p$$

$$-kx\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$N = p$$

$$-kx\,\mathrm{d}x = mv\,\mathrm{d}v$$

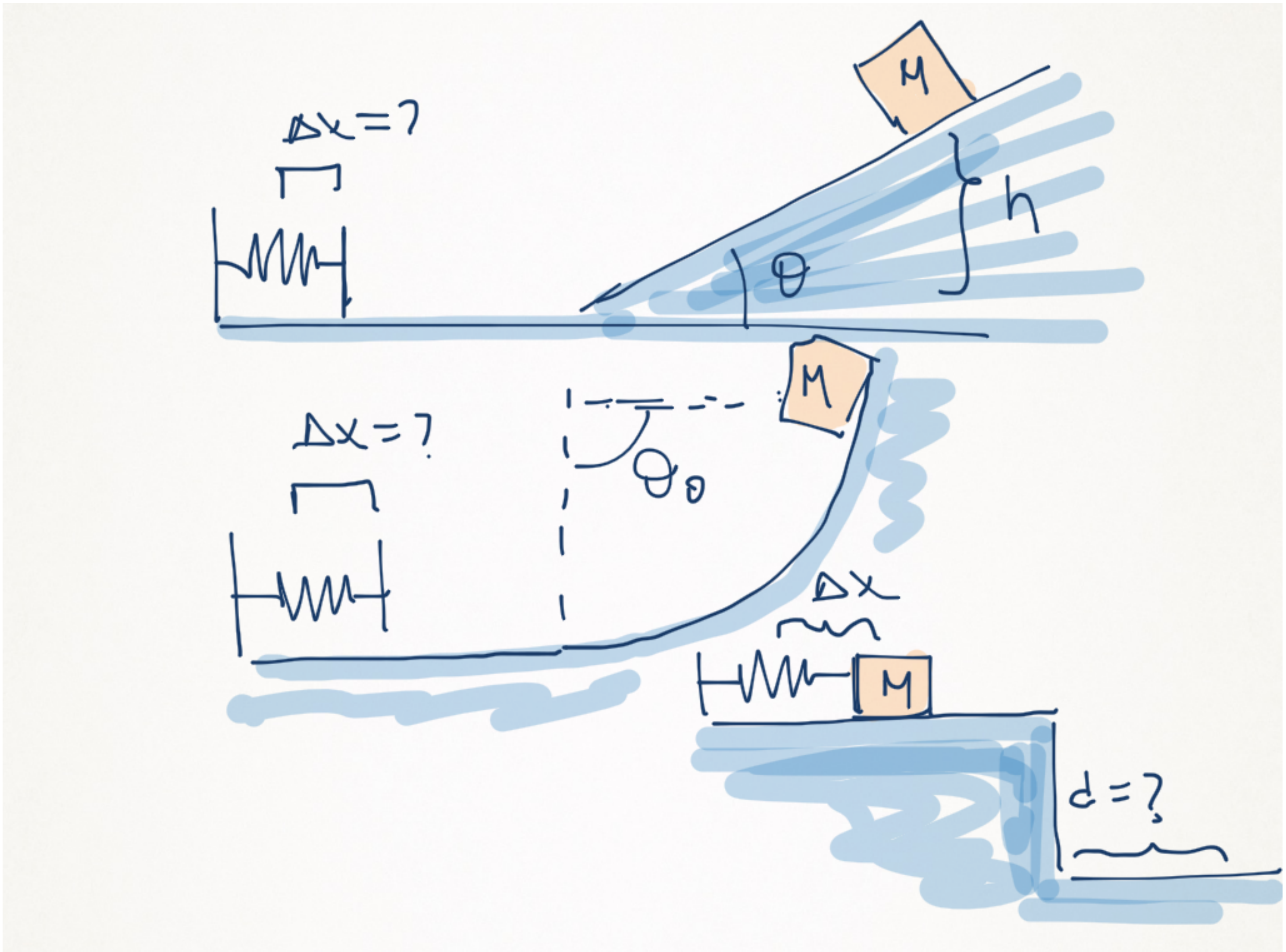
$$N = p$$

$$-\int_{-x_c}^{0} kx\,\mathrm{d}x = m\int_{0}^{v_f} v\,\mathrm{d}v$$

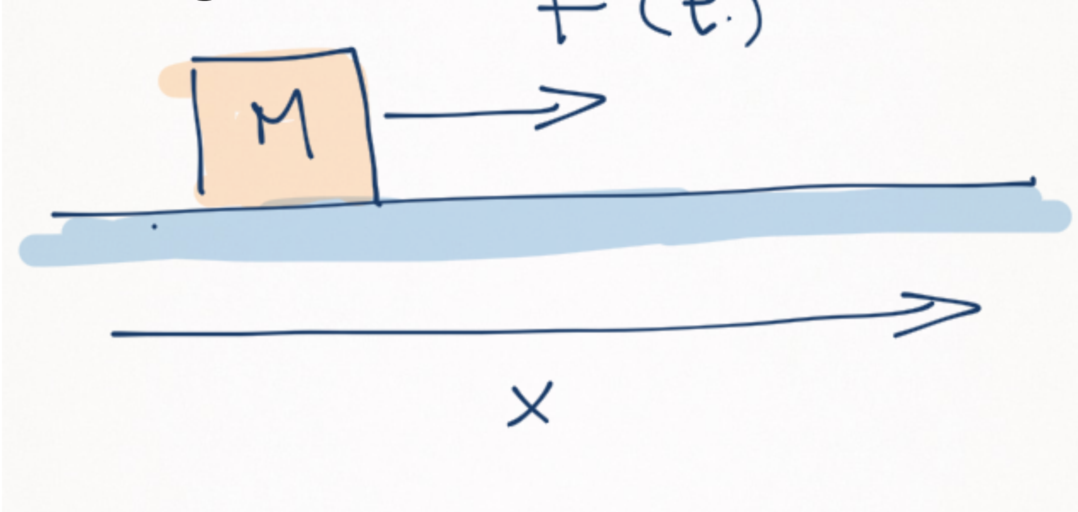
$$\frac{1}{2}kx_c^2 = \frac{1}{2}mv_f^2$$

$$v_f = x_c\sqrt{\frac{k}{m}}$$

Situaciones varias



En general



$$F(t) = ma$$

$$F(t) = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$F(t)\mathrm{d}t = m\mathrm{d}v$$

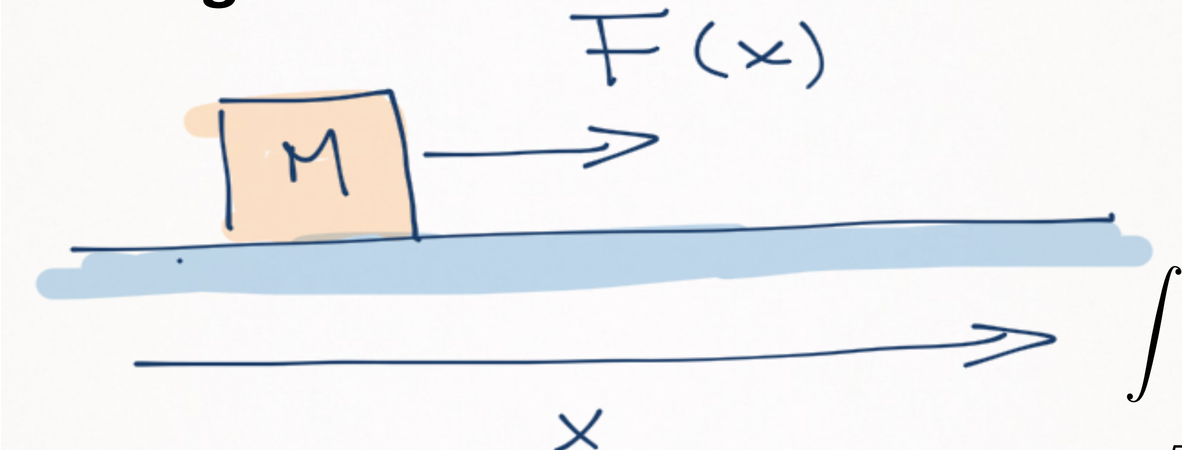
$$\int^{t} F(\tilde{t})\,\mathrm{d}\tilde{t} + C_1 = mv$$

$$\int^{t} F(\tilde{t})\,\mathrm{d}\tilde{t} + C_1 = m\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\int^{t}\left(\int^{\bar{t}} F(\tilde{t})\,\mathrm{d}\tilde{t} + C_1\right)\mathrm{d}\bar{t} + C_2 = m\int^{t}\mathrm{d}x$$

$$\int^{t}\left(\int^{\bar{t}} F(\tilde{t})\,\mathrm{d}\tilde{t} + C_1\right)\mathrm{d}\bar{t} + C_2 = mx(t)$$

En general



$$F(x) = ma$$

$$F(x) = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$F(x)\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}t}v$$

$$\int F(\tilde{x})\,\mathrm{d}\tilde{x} + C_1 = m\int v\,\mathrm{d}v$$

$$\sqrt{\frac{2}{m}}\sqrt{\int^x F(\tilde{x})\,\mathrm{d}\tilde{x} + C_1} = v$$

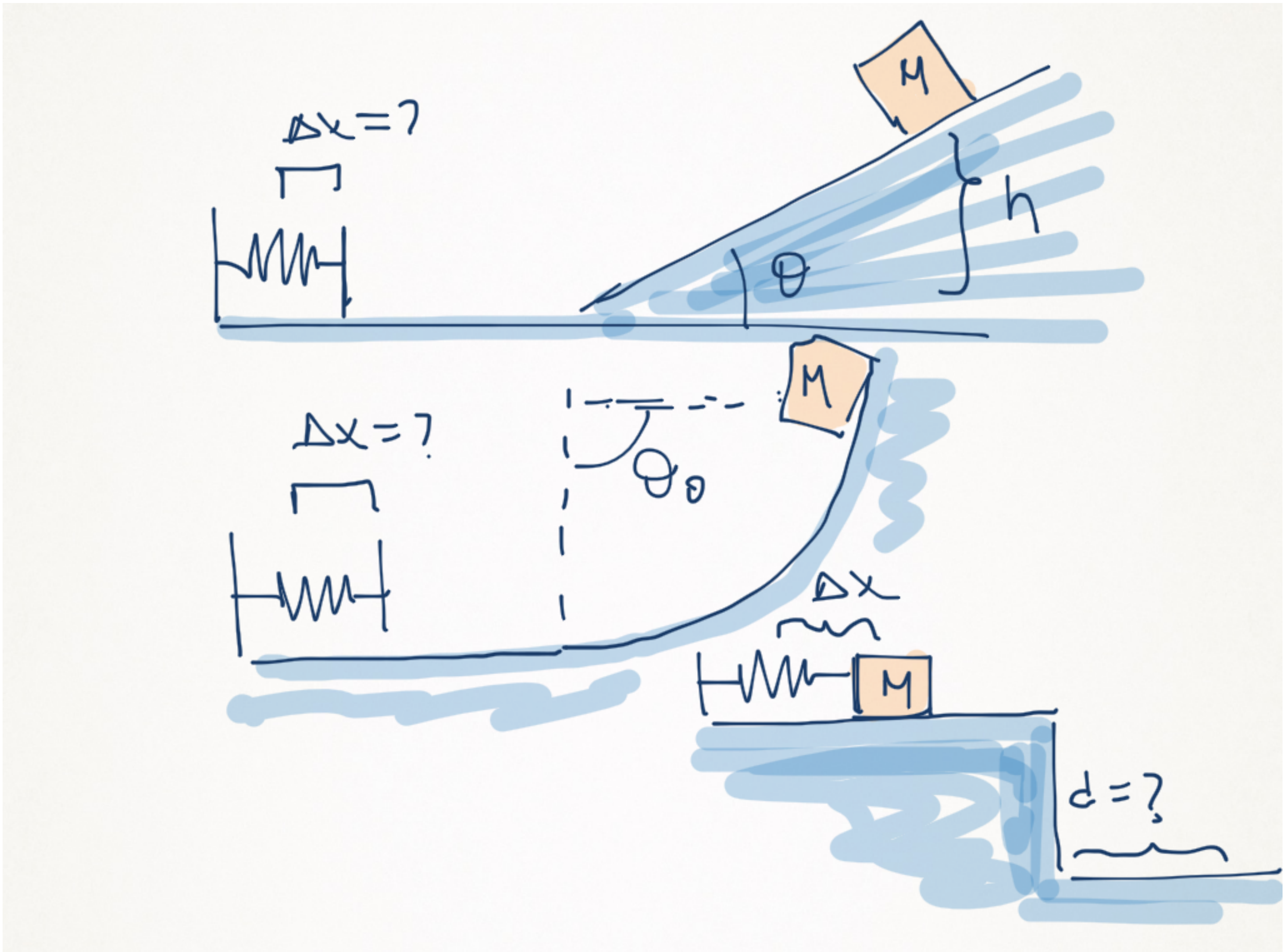
$$\sqrt{\frac{2}{m}}\sqrt{\int^x F(\tilde{x})\,\mathrm{d}\tilde{x} + C_1} = \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\int_{t_0}^{t}\mathrm{d}\tilde{t} = \int_{x_0}^{x}\frac{\mathrm{d}\bar{x}}{\sqrt{\frac{2}{m}}\sqrt{\int^{\bar{x}} F(\tilde{x})\,\mathrm{d}\tilde{x} + C_1}} + C_2$$

$$t - t_0 = \mathcal{F}(x, C_1, C_2)$$

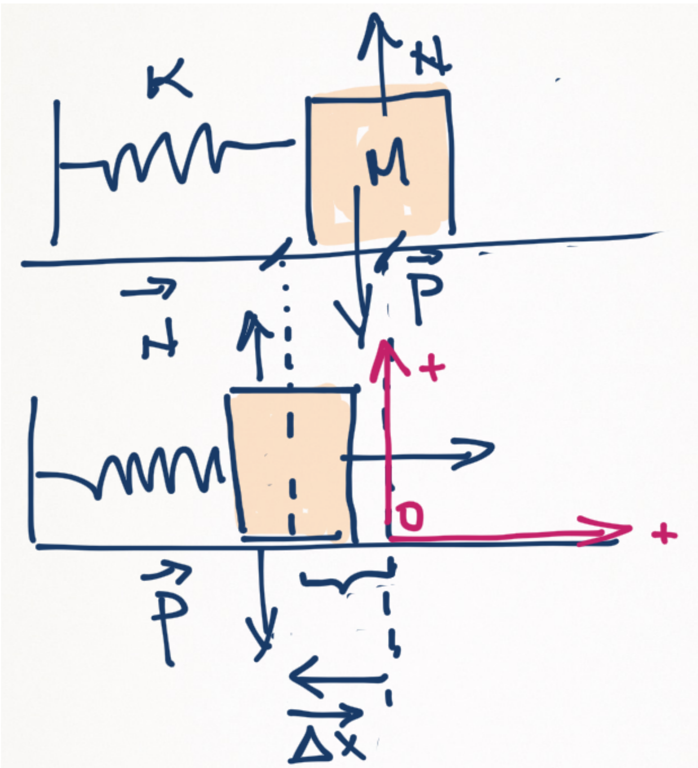
$$x \stackrel{?}{=} x(t, C_1, C_2)$$

Situaciones varias



Escuela de Física | Universidad industrial de Santander | Grupo Halley Astronomía y Ciencias Aeroespaciales | Complejo Astronómico Grupo Halley

Fuerzas elásticas y las ecuaciones de movimiento



$$-\frac{k}{m}x(t) = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}$$

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \omega^2 x(t) = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$x(t) = ? \quad \text{con} \begin{cases} x(0) = x_0 \\ v(0) = \left.\frac{\mathrm{d}v(t)}{\mathrm{d}t}\right|_{t=0} \end{cases}$$

```
(%i1) ecuacDif1: %omega^2*x(t)+diff(x(t),t,2);assume(%omega^2 >0);
```

(%o1) $\frac{d^2}{d\,t^2}x(t)+\omega^2 x(t)$

(%o2) $[\omega^2>0]$

```
(%i3) desolve(ecuacDif1,x(t));
```

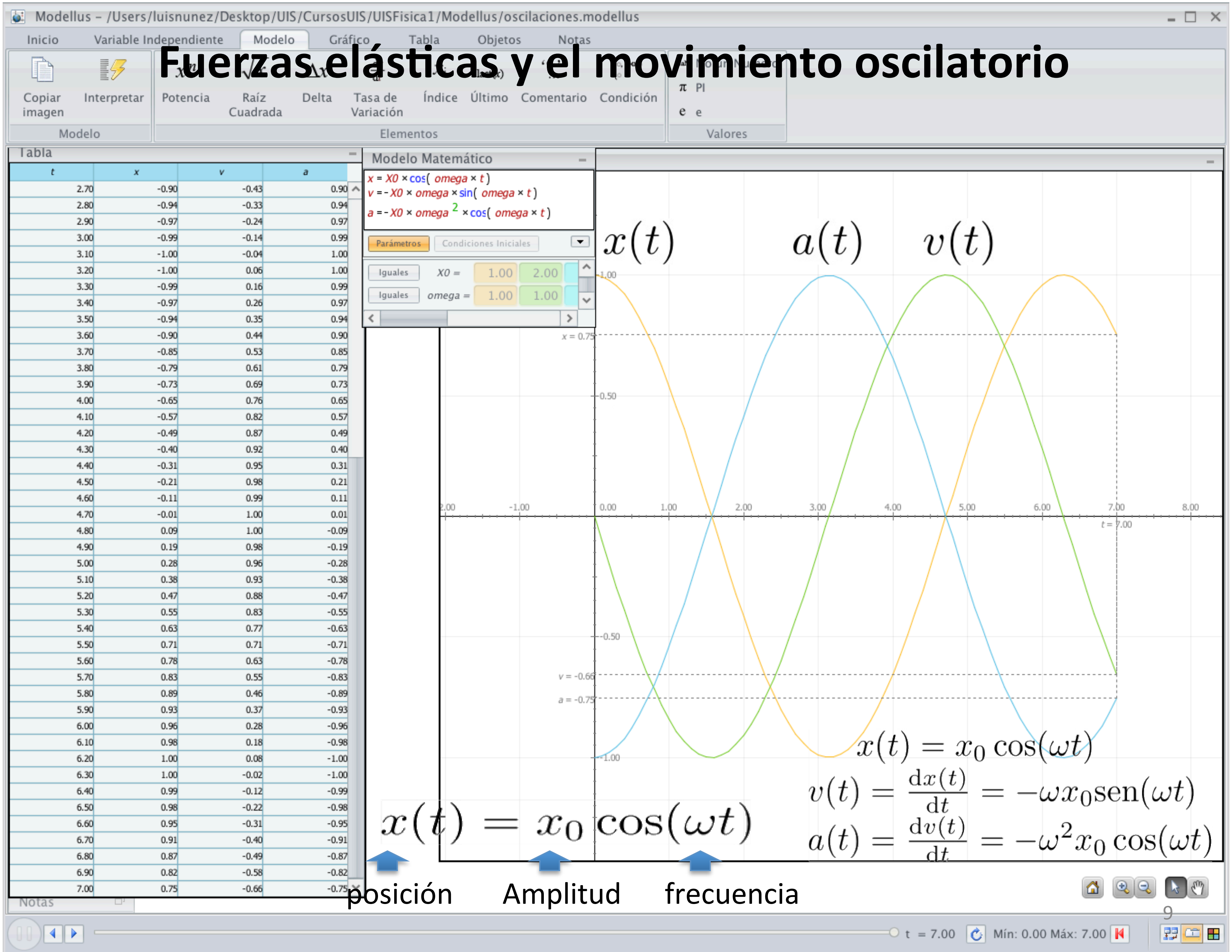
(%o3) $x(t)=\frac{\sin(\omega\, t)\left(\left.\frac{d}{d\,t}x(t)\right|_{t=0}\right)}{\omega}+x(0)\cos(\omega\, t)$

$$x(t) = x_0 \cos(\omega t)$$

$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\omega x_0 \mathrm{sen}(\omega t)$$

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\omega^2 x_0 \cos(\omega t)$$

Fuerzas elásticas y el movimiento oscilatorio



t	x	v	a
2.70	-0.90	-0.43	0.90
2.80	-0.94	-0.33	0.94
2.90	-0.97	-0.24	0.97
3.00	-0.99	-0.14	0.99
3.10	-1.00	-0.04	1.00
3.20	-1.00	0.06	1.00
3.30	-0.99	0.16	0.99
3.40	-0.97	0.26	0.97
3.50	-0.94	0.35	0.94
3.60	-0.90	0.44	0.90
3.70	-0.85	0.53	0.85
3.80	-0.79	0.61	0.79
3.90	-0.73	0.69	0.73
4.00	-0.65	0.76	0.65
4.10	-0.57	0.82	0.57
4.20	-0.49	0.87	0.49
4.30	-0.40	0.92	0.40
4.40	-0.31	0.95	0.31
4.50	-0.21	0.98	0.21
4.60	-0.11	0.99	0.11
4.70	-0.01	1.00	0.01
4.80	0.09	1.00	-0.09
4.90	0.19	0.98	-0.19
5.00	0.28	0.96	-0.28
5.10	0.38	0.93	-0.38
5.20	0.47	0.88	-0.47
5.30	0.55	0.83	-0.55
5.40	0.63	0.77	-0.63
5.50	0.71	0.71	-0.71
5.60	0.78	0.63	-0.78
5.70	0.83	0.55	-0.83
5.80	0.89	0.46	-0.89
5.90	0.93	0.37	-0.93
6.00	0.96	0.28	-0.96
6.10	0.98	0.18	-0.98
6.20	1.00	0.08	-1.00
6.30	1.00	-0.02	-1.00
6.40	0.99	-0.12	-0.99
6.50	0.98	-0.22	-0.98
6.60	0.95	-0.31	-0.95
6.70	0.91	-0.40	-0.91
6.80	0.87	-0.49	-0.87
6.90	0.82	-0.58	-0.82
7.00	0.75	-0.66	-0.75

$$x(t) = x_0 \cos(\omega t)$$

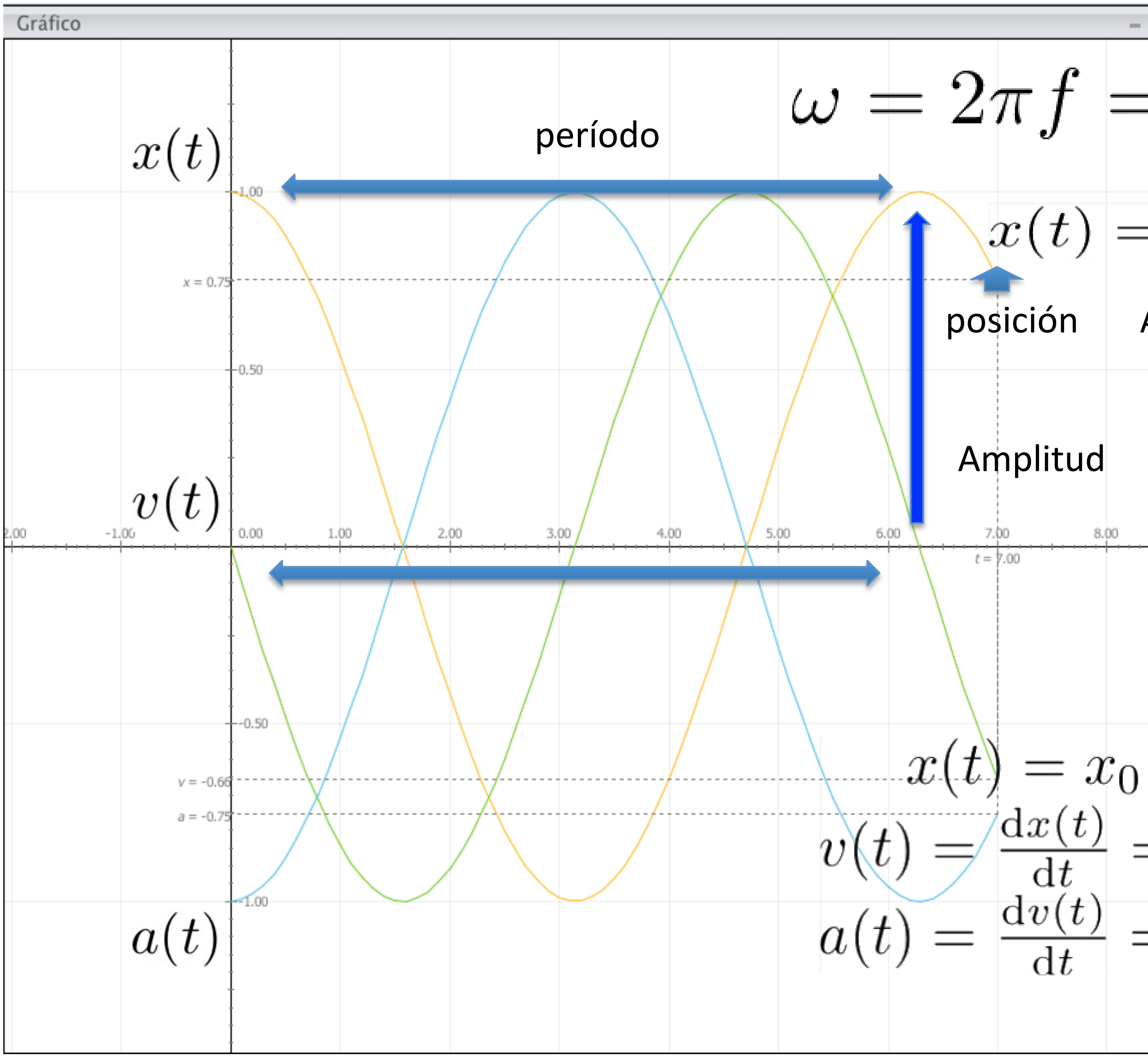
$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\omega x_0 \mathrm{sen}(\omega t)$$

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\omega^2 x_0 \cos(\omega t)$$

$$x(t) = x_0 \cos(\omega t)$$

posición Amplitud frecuencia

Fuerzas elásticas: período y frecuencia



$$\omega = 2\pi f = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$x(t) = x_0 \cos(\omega t)$$

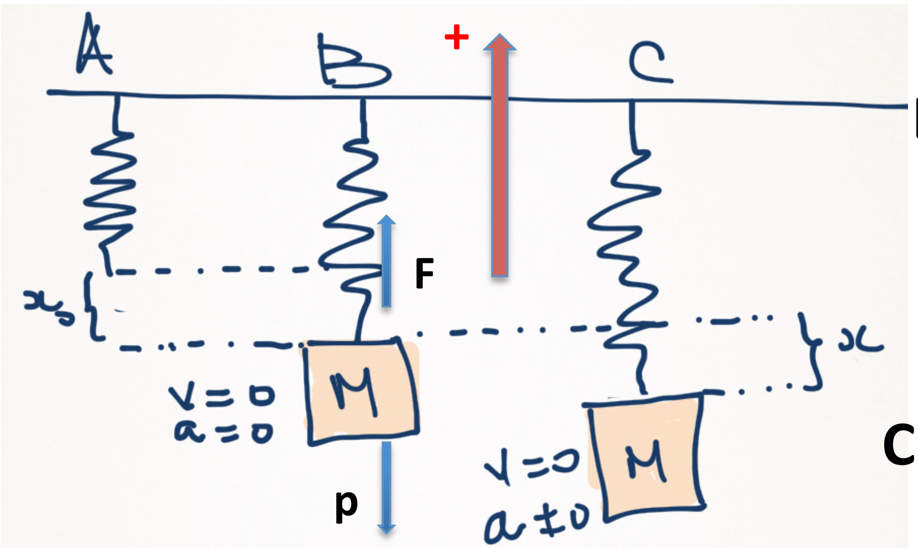
posición — Amplitud — frecuencia

$$x(t) = x_0 \cos(\omega t)$$

$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\omega x_0 \mathrm{sen}(\omega t)$$

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\omega^2 x_0 \cos(\omega t)$$

Fuerzas elásticas y las ecuaciones de movimiento



B)

$$\vec{F} + \vec{p} = 0 \qquad \Rightarrow F - p = 0$$

$$ky_0 = p \qquad \Rightarrow y_0 = \frac{p}{k}$$

C)

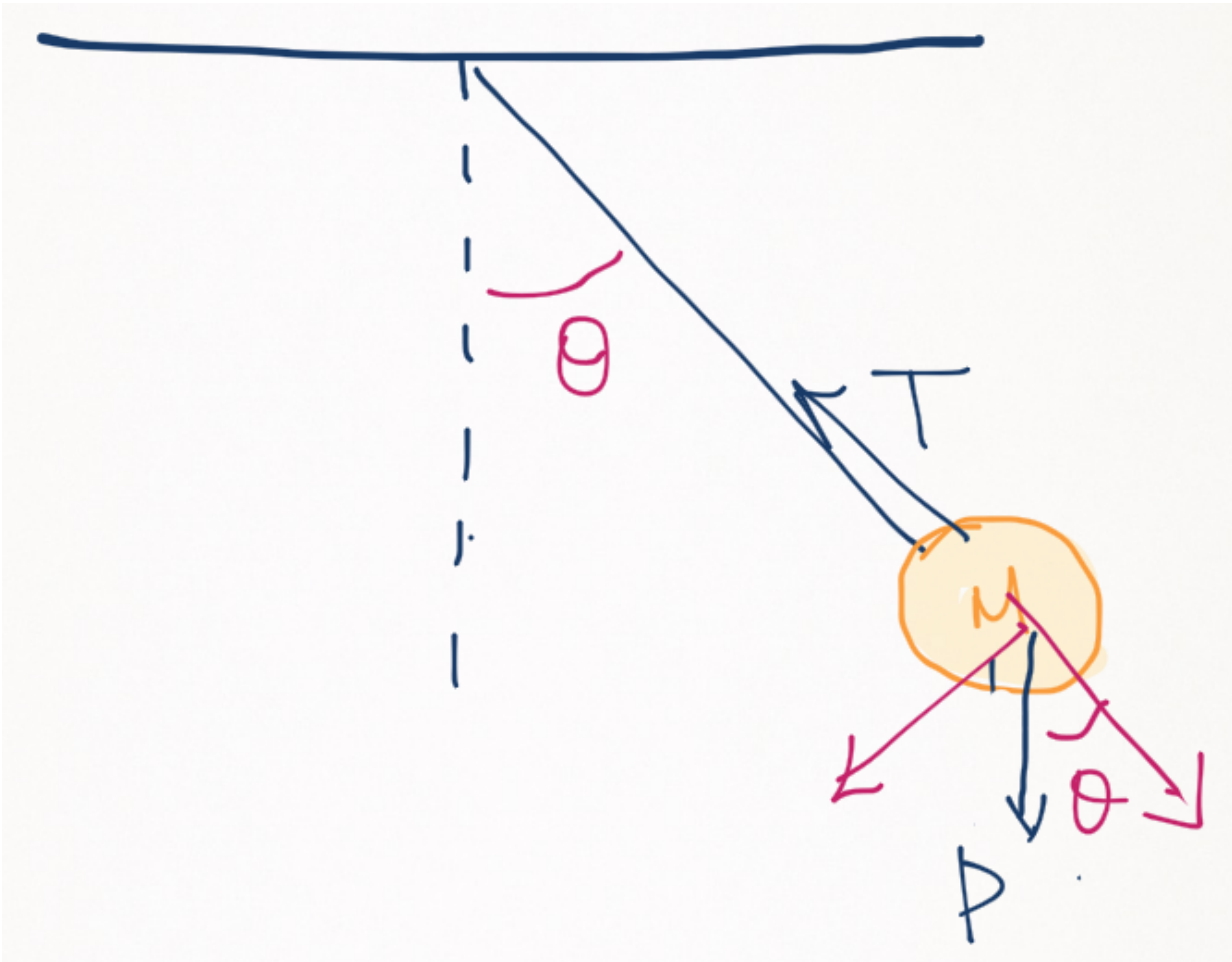
$$\vec{F} + \vec{p} = m\vec{a} \qquad \Rightarrow F - p = ma$$

$$k(-y(t) + y_0) - p = ma$$

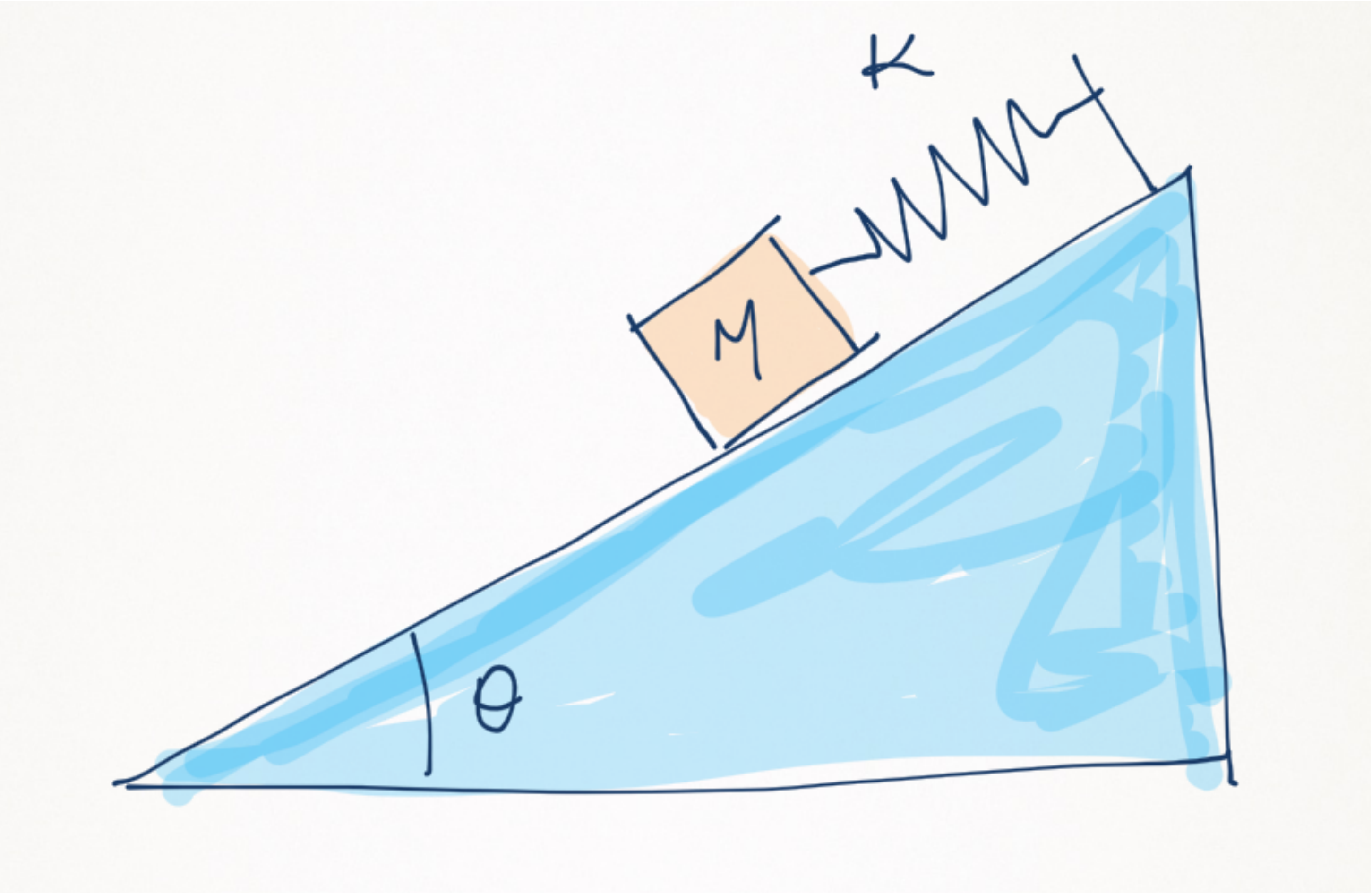
$$\Rightarrow -ky(t) = m\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}$$

$$\Rightarrow m\ddot{y}(t) + ky(t) = 0$$

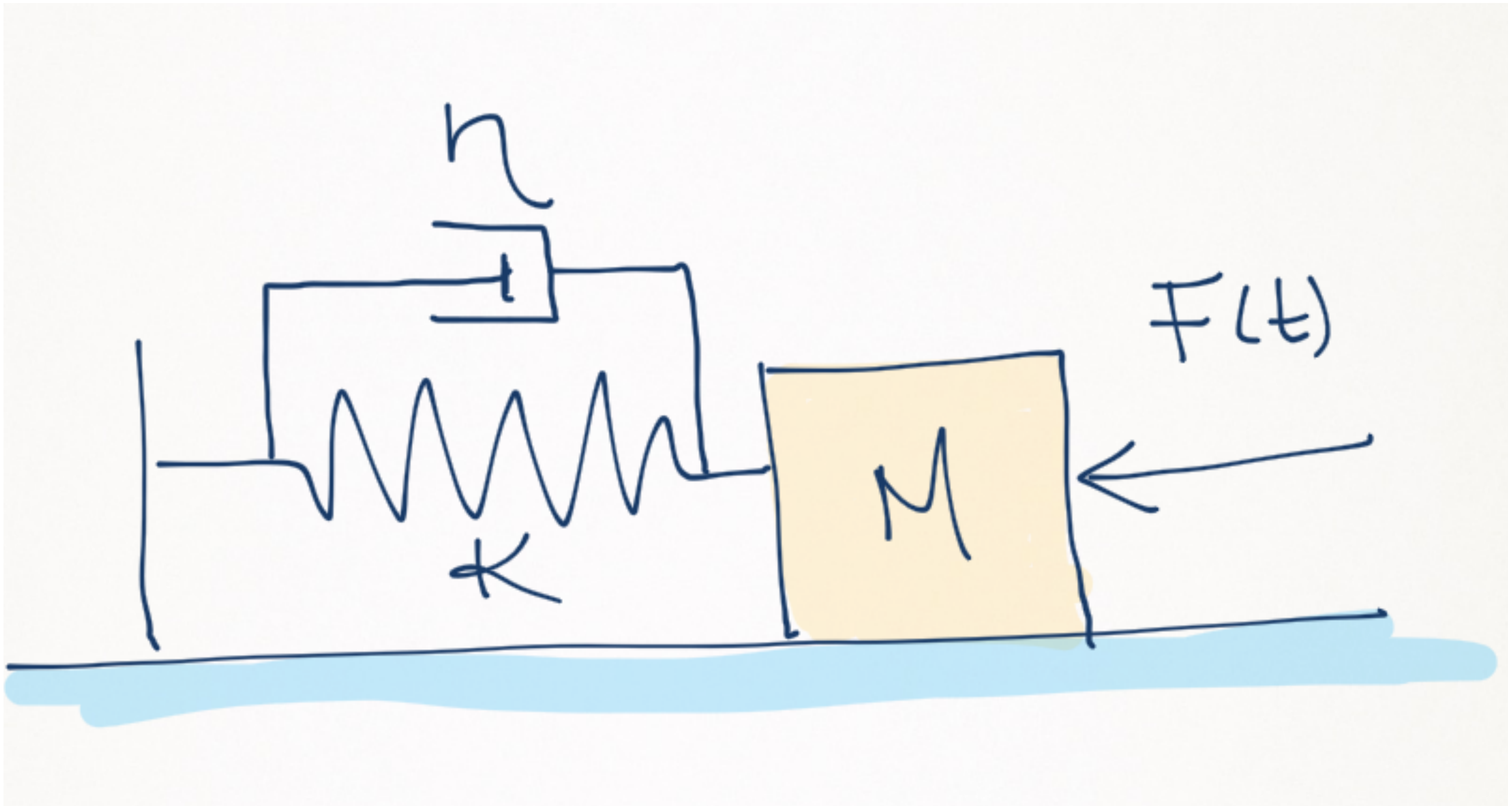
Más oscilaciones



Más oscilaciones



En general



En general

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\eta\,\frac{\mathrm{d}x}{\mathrm{d}t}+k\,x=F\left(t\right)$$

donde

x	$\Rightarrow$	Desplazamiento
$\frac{\mathrm{d}x}{\mathrm{d}t}$	$\Rightarrow$	Velocidad
m	$\Rightarrow$	masa
η	$\Rightarrow$	Constante de Amortiguamientc
k	$\Rightarrow$	Constante ElÃ¡stica
$F\left(t\right)$	$\Rightarrow$	Fuerza Aplicada

$$L\,\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}+R\,\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{1}{C}\,Q=E\left(t\right)$$

donde

Q	$\Rightarrow$	Carga Eléctrica
$\frac{\mathrm{d}Q}{\mathrm{d}t}=I$	$\Rightarrow$	Intensidad de Corriente
L	$\Rightarrow$	Inductancia
R	$\Rightarrow$	Resistencia
C	$\Rightarrow$	Capacitancia
$E\left(t\right)$	$\Rightarrow$	Fuerza Electromotriz

$$\alpha\;\ddot{u}+\beta\;\dot{u}+\gamma\;u\equiv\alpha\;\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\beta\;\frac{\mathrm{d}u}{\mathrm{d}t}+\gamma\;u=\Lambda\left(t\right)$$

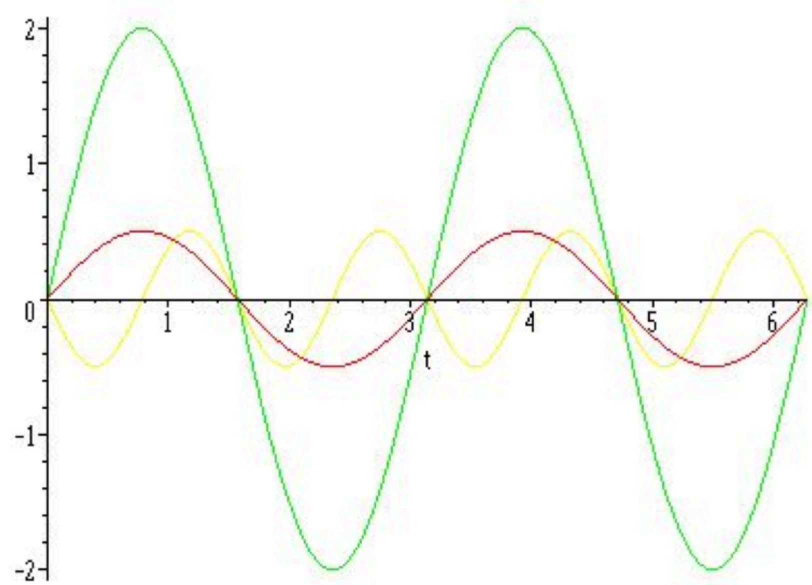
Oscilaciones Libres

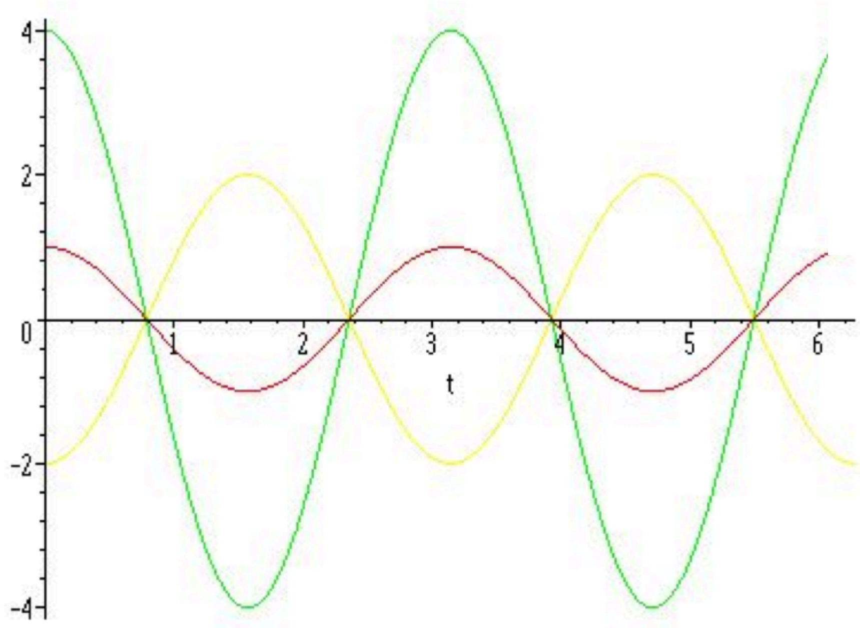
$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+k\;x=0 \qquad \Rightarrow \qquad x\left(t\right)=C_1\cos\left(\omega_0 t\right)+C_2\,\mathrm{sen}\left(\omega_0 t\right) \qquad \text{con} \quad \omega_0=\sqrt{\frac{k}{m}}$$

$$\text{si} \quad \left\{\begin{array}{l} C_1=A\cos\delta \\ C_2=A\,\mathrm{sen}\,\delta \end{array}\right. \quad \Rightarrow \quad x\left(t\right)=C_1\cos\left(\omega_0 t\right)+C_2\,\mathrm{sen}\left(\omega_0 t\right) \quad \Leftrightarrow \quad x\left(t\right)=A\cos\left(\omega_0 t+\delta\right)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+4\;x=0 \quad \wedge \quad \left\{\begin{array}{lll} x\left(0\right)=1; \quad \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0}=0; & \Rightarrow & x\left(t\right)=\cos(2t) \\ x\left(0\right)=4; \quad \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0}=0 & \Rightarrow & x\left(t\right)=4\cos\left(2t\right) \\ x\left(0\right)=-2; \quad \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0}=0 & \Rightarrow & x\left(t\right)=-2\cos\left(2t\right) \end{array}\right.$$

Oscilaciones libres





Oscilador armónico libre. Cambios en la posición inicial no afectan la frecuencia natural.

Oscilaciones libres amortiguadas

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\eta\,\frac{\mathrm{d}x}{\mathrm{d}t}+k\,x=0\quad\Leftrightarrow\quad\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\mu\,\frac{\mathrm{d}x}{\mathrm{d}t}+\omega_0^2\,x=0$$

$$x\left(t\right)=C_1e^{\left(-\left(\mu+\sqrt{\mu^2-\omega_0^2}\right)t\right)}+C_2e^{\left(-\left(\mu-\sqrt{\mu^2-\omega_0^2}\right)t\right)}$$

$$x\left(t\right)=C_1\;e^{r_1t}+C_2\;e^{r_2t}\qquad\Leftarrow\quad\mu^2-\omega_0^2>0\quad\text{Sobreamortiguado}$$

$$x\left(t\right)=\left(C_1+C_2\;t\right)e^{\mu\;t}\qquad\Leftarrow\quad\mu^2-\omega_0^2=0\quad\text{Crítico}$$

$$x\left(t\right)=e^{-\mu\;t}\left\{C_1\cos\left[\left(\sqrt{\omega_0^2-\mu^2}\right)t\right]+C_2\,\mathrm{sen}\left[\left(\sqrt{\omega_0^2-\mu^2}\right)t\right]\right\}\quad\Leftarrow\quad\mu^2-\omega_0^2<0\quad\text{Subamortiguado}$$

Oscilaciones libres amortiguadas

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\,\frac{\mathrm{d}x}{\mathrm{d}t} + 4\,x = 0 \quad \wedge \quad \left\{ \begin{array}{l} x(0) = 0 \\ \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = 4 \end{array} \right\} \Rightarrow x(t) = \left(\frac{1}{2} + \frac{7}{2\sqrt{5}}\right) e^{\left(\sqrt{5}-3\right)t} + \left(\frac{1}{2} - \frac{7}{2\sqrt{5}}\right) e^{-\left(3+\sqrt{5}\right)t}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\,\frac{\mathrm{d}x}{\mathrm{d}t} + 4\,x = 0 \quad \wedge \quad \left\{ \begin{array}{l} x(0) = 0 \\ \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = 4 \end{array} \right\} \Rightarrow x(t) = (1 + 6t)\, e^{-2t}$$

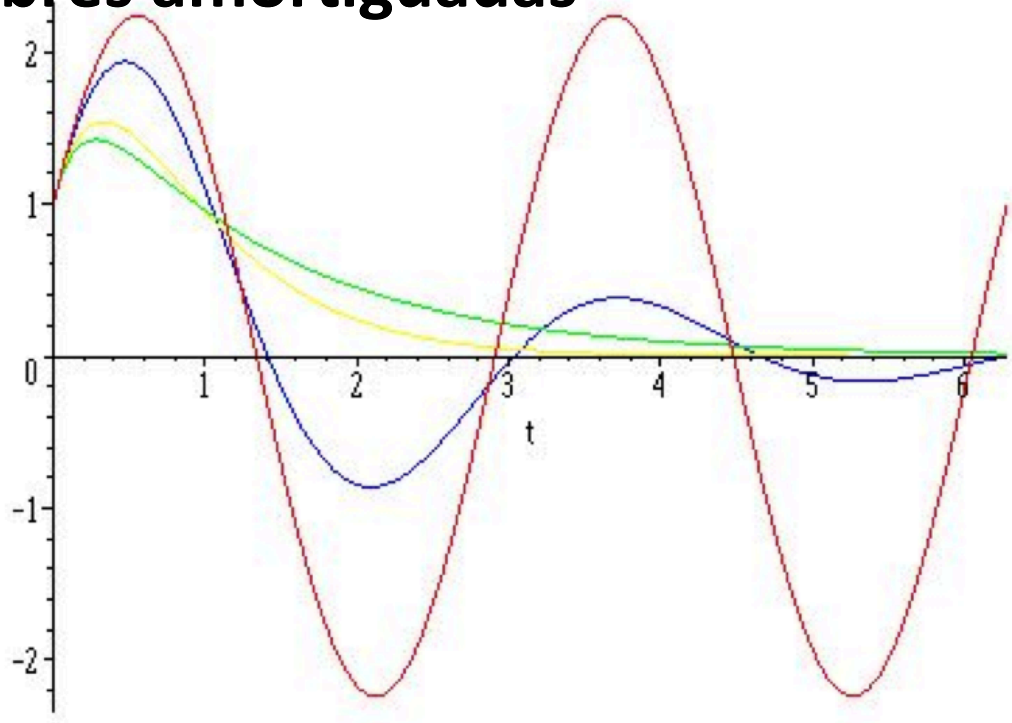
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} + 4\,x = 0 \quad \wedge \quad \left\{ \begin{array}{l} x(0) = 0 \\ \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = 4 \end{array} \right\} \Rightarrow x(t) = e^{-\frac{1}{2}t}\left[\frac{9}{\sqrt{15}}\,\mathrm{sen}\left(\frac{\sqrt{15}}{2}t\right) + \cos\left(\frac{\sqrt{15}}{2}t\right)\right]$$

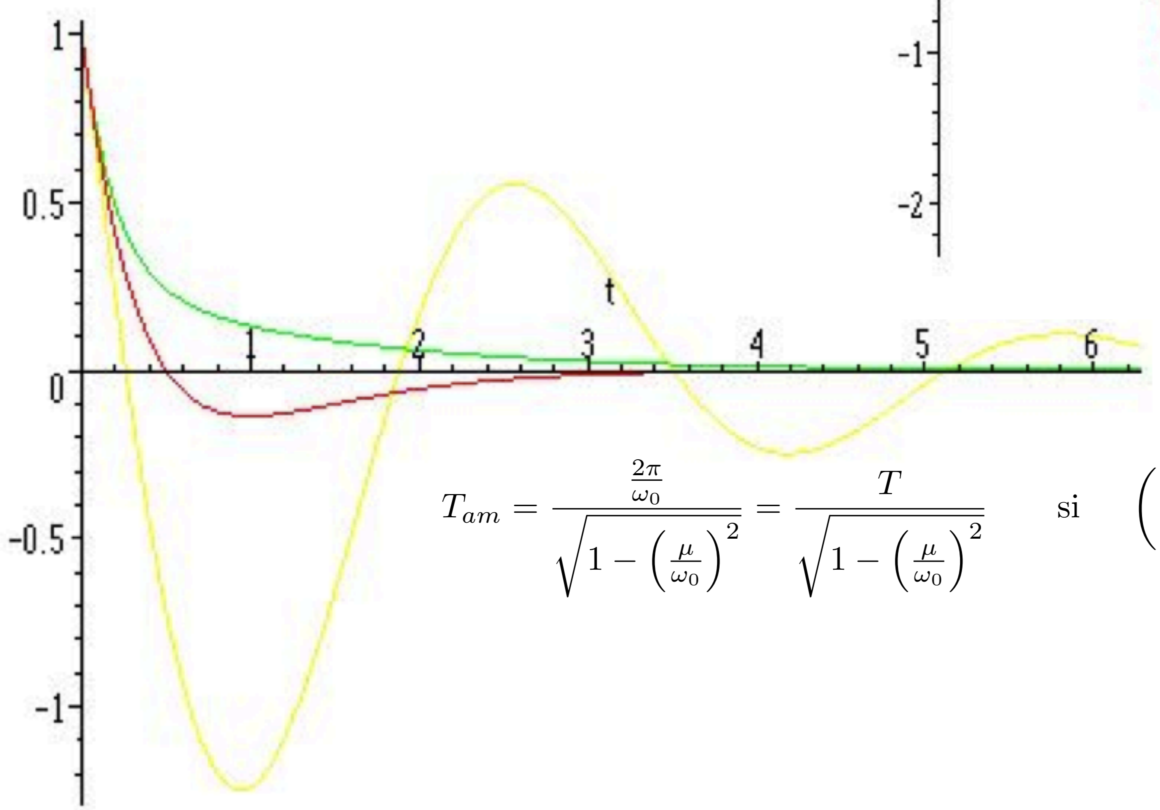
$$x(0) = 1; \quad \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = -4; \quad \Rightarrow \quad x(t) = \left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right) e^{\left(\sqrt{5}-3\right)t} + \left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right) e^{-\left(3+\sqrt{5}\right)t}$$

$$x(0) = 1; \quad \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = -4; \quad \Rightarrow \quad x(t) = (1 + 2t)\, e^{-2t}$$

$$x(0) = 1; \quad \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = -4 \quad \Rightarrow \quad x(t) = e^{-\frac{1}{2}t}\left[\frac{-7}{\sqrt{15}}\,\mathrm{sen}\left(\frac{\sqrt{15}}{2}t\right) + \cos\left(\frac{\sqrt{15}}{2}t\right)\right]$$

Oscilaciones libres amortiguadas



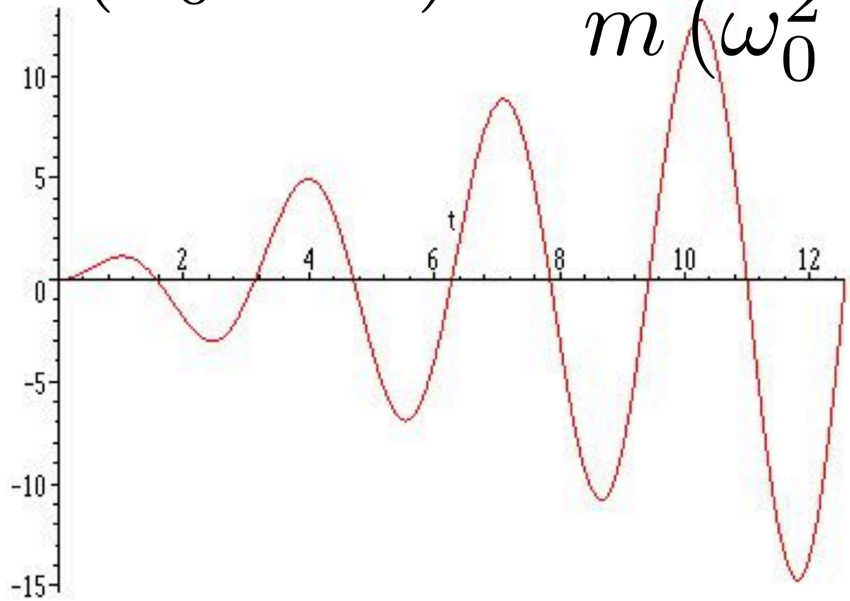


$$T_{am} = \frac{\frac{2\pi}{\omega_0}}{\sqrt{1-\left(\frac{\mu}{\omega_0}\right)^2}} = \frac{T}{\sqrt{1-\left(\frac{\mu}{\omega_0}\right)^2}} \qquad \text{si} \quad \left(\frac{\mu}{\omega_0}\right)^2 << 1 \quad \Rightarrow \quad T_{am} \approx T\left(1+\frac{1}{2}\left(\frac{\mu}{\omega_0}\right)^2\right)$$

Oscilaciones forzadas

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_0^2 \; x = \frac{F_0}{m} \cos\left(\varpi t\right)$$

$$x\left(t\right) = A\cos\left(\omega_0 t + \delta\right) + \frac{F_0}{m\left(\omega_0^2 - \varpi^2\right)} \cos\left(\varpi t\right)$$



Oscilador armónico forzado con $\varpi = \omega_0^2$ Nótese el fenómeno de resonancia